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THE VIRTUAL MASS AND LIFT FORCE ON A SPHERE IN ROTATING AND STRAINING INVISCID FLOW

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1. INTRODUCTION

A single spherical particle moving through a fluid experiences forces which affect its the motion. If the relative velocity between the fluid and the particle is constant, the force is usually called the drag. Such forces have been intensively studied by Zuber & Ishii (1978). When the motion is unsteady, or nonuniform it is natural to attempt to extend the concept of a drag force to include the various nondrag forces which arise. Typical nondrag forces include the so-called virtual mass force and the lateral lift force.

The original calculation of the so-called virtual mass force is attributed to Lord Kelvin (Lamb 1932). A sphere accelerating through a quiescent fluid experiences a resistance force proportional to its acceleration. For a rigid sphere accelerating through an inviscid fluid of constant density, the constant of proportionality is one-half the mass of the fluid displaced by the sphere.

The force on a sphere moving relative to a rotating fluid has been previously calculated by Proudman (1916). This force is perpendicular to the rotation vector and the sphere velocity vector, and represents the classical lift force. In situations where the Coriolis force dominates, the fluid motion is dominated by a Taylor column (Greenspan 1968).

If the problem of describing a dispersion of particles in a fluid is approached by deriving equations for the motion of two interpenetrating continua, it becomes necessary to specify the interaction force density as a function of the average flow fields. One of the principles employed in such an approach is that of objectivity (Drew & Lahey 1979), or material frame indifference, which implies that the interaction forces should be independent of the coordinate system used to describe them. This principle has been criticized recently (Ryskin & Rallison 1980; Micaelli 1983; Auton 1983); the most serious criticism being that the objective form of the virtual mass force did not reduce to that calculated for a single sphere (Voinov 1978). It is the purpose of this paper to resolve this controversy.

In section 2, we derive the force on a single sphere accelerating relative to an inviscid fluid undergoing a pure strain and rotation far from the sphere. This inviscid force implicitly assumes that the viscous boundary layer on the sphere does not separate from the sphere. This precludes the formation of Taylor columns (Greenspan 1968). The net force on the sphere consists of three parts: the pressure gradient force, and a nondrag force which consists of the virtual mass force and the lift force. The nondrag force is shown to be objective and to agree with the invarient force previously postulated (Drew & Lahey 1979).

2. FORCE ON A SPHERE IN LINEAR SHEAR FLOW

Consider a sphere moving through an inviscid fluid with velocity components v^*_{di}. The fluid far from the sphere is assumed to be undergoing a motion which consists of unsteady translation, a rotation at constant angular velocity and a constant strain. The velocity of the fluid far from the sphere is thus (Aris 1962)

$$v_{ci}^* = v_{0i}^* + x_j^* v_{ci,j}^*$$

or

$$v_{ci}^* = v_{0i}^* + e_{ij}^* x_j^* + \epsilon_{ijk}\omega_j^* x_k^*, \qquad [1]$$

where x_i^* is the spatial coordinate, v_{0i}^* is the (undisturbed) fluid velocity at the origin, $e_{ij}^* = \frac{1}{2}(v_{ci,j}^* + v_{cj,i}^*)$ is the symmetric tensor representing the straining motion and ω_i^* is the angular rotation vector, which satisfies $\epsilon_{ijk}\omega_j^* = \frac{1}{2}(v_{ci,k}^* - v_{ck,i}^*)$. Since the fluid is incompressible, continuity implies

$$e_{ii}^* = 0. \qquad [2]$$

Let us now consider a noninertial (unstarred) coordinate system which rotates with the fluid far from the sphere. Then the new coordinate system is related to the previous one by

$$x_i = Q_{ij}x_j^*, \qquad [3]$$

where Q_{ij} is the orthonormal rotation tensor which satisfies

$$Q_{ij}Q_{kj} = \delta_{ik} = Q_{ji}Q_{jk}. \qquad [4]$$

We further assume that Q_{ij} is proper, so that det $[Q_{ij}] = 1$.

The velocity of the fluid in the unstarred coordinate frame is

$$v_{ci} = \dot{x}_i = Q_{ij}v_{cj}^* + \dot{Q}_{ij}x_j^*$$

or

$$v_{ci} = Q_{ij}v_{0j}^* + Q_{ij}e_{jk}^*x_k^* + Q_{ij}\epsilon_{jkl}\omega_k^*x_l^* + \dot{Q}_{ij}x_j^*. \qquad [5]$$

If we take

$$\dot{Q}_{ij} = -Q_{ik}\epsilon_{kmj}\omega_m^*, \qquad [6]$$

and define

$$e_{ij} = Q_{ik}Q_{jl}e_{kl}^* \qquad [7]$$

and

$$v_{0i} = Q_{ij}v_{0j}^*, \qquad [8]$$

then

$$\begin{aligned} v_{ci} &= v_{0i} + e_{ij}x_j \\ &= Q_{ij}v_{cj}^*. \end{aligned} \qquad [9]$$

Without loss of generality, we can assume that the rotation vector is orientated such that

$$\omega_i^* = \omega\delta_{i3}. \qquad [10]$$

The velocity of the sphere in the unstarred coordinate frame is given by

$$v_{di} = Q_{ij}v_{dj}^* + \dot{Q}_{ij}x_j^*. \qquad [11]$$

In the rotating (unstarred) coordinate frame the equations of motion for the flow of interest here are (Truesdell & Toupin 1963)

$$v_{i,i} = 0 \qquad [12]$$

and

$$v_{i,t} - \epsilon_{ijk}v_j\zeta_k + 2\epsilon_{ijk}\omega_jv_k = -P_{,i}. \qquad [13]$$

Here $\zeta_i = \epsilon_{ijk}v_{k,j}$ is the vorticity and

$$P = \frac{p}{\rho_c} + \tfrac{1}{2}v_iv_i - \tfrac{1}{2}\omega^2(x_1^2 + x_2^2), \qquad [14]$$

where p is the static pressure and ρ_c is the density of the fluid.

It is well known that if the flow is inviscid and irrotational at some initial instant, then it will remain irrotational. This was first noted by Proudman (1916) for the situation where $e_{ij} = 0$ and v_{0i} was constant. Thus, we seek a solution with

$$\zeta_i = \epsilon_{ijk} v_{k,j} = 0. \qquad [15]$$

This implies the existence of a velocity potential ϕ, such that

$$v_i = \phi_{,i}. \qquad [16]$$

Thus, the continuity equation [12] implies

$$\phi_{,ii} = 0 \qquad [17]$$

outside the sphere. The boundary condition on the surface of the sphere is

$$v_i n_i = \phi_{,i} n_i = v_{di} n_i, \qquad [18]$$

where n_i is the unit normal at the surface of the sphere; thus,

$$n_i = \frac{(x_i - x_{di})}{R}, \qquad [19]$$

where R is the radius of the sphere. Far from the sphere the velocity should approach the undisturbed fluid velocity, v_{ci}, thus

$$\lim_{(x_i - x_{di}) \to \infty} \phi_{,i} = v_{0i} + e_{ij} x_j. \qquad [20]$$

Using [15] in [13], taking the divergence of the result and using [17], we have

$$P_{,ii} = 0 \qquad [21]$$

in the fluid exterior to the sphere. At the surface of the sphere, we have

$$-n_i P_{,i} = n_i \phi_{,it} + 2 n_i \epsilon_{ijk} \omega_j \phi_{,k}, \qquad [22]$$

while far from the sphere P satisfies

$$\lim_{(x_i - x_{di}) \to \infty} P_{,i} = -v_{0i,t} - 2\epsilon_{ijk}\, \omega_j v_{ck}. \qquad [23]$$

Thus, P can be written as

$$P = -v_{0i,t}(x_i - x_{di}) - 2(x_i - x_{di})\epsilon_{ijk}\omega_j v_{0k} - 2(x_i - x_{di})\epsilon_{ijk}\omega_j E_{kl}(x_l - x_{dl}) + P', \qquad [24]$$

where

$$E_{ij} = e_{ij} \quad (i \neq j)$$

and

$$E_{ii} = \frac{e_{ii}}{2} \quad \text{(no sum on } i\text{)}, \qquad [25]$$

and P' satisfies

$$P'_{,ii} = 0, \qquad [26]$$

$$\lim_{(x_i - x_{id}) \to \infty} P'_{,i} = 0, \qquad [27]$$

in the region exterior to the sphere, and

$$-n_i P'_{,i} = n_i(\phi_{,it} - v_{0i,t}) + 2 n_i \epsilon_{ijk} \omega_j (\phi_{,k} - v_{ck}) \qquad [28]$$

on the surface of the sphere.

The net force on the sphere is given by [14] as

$$-\iint n_i p \, dS = \tfrac{1}{2}\rho_c \iint v_j v_j n_i \, dS - \tfrac{1}{2}\rho_c \omega^2 \iint (x_1^2 + x_2^2) n_i \, dS - \rho_c \iint P n_i \, dS. \qquad [29]$$

From [24], we have

$$-\rho_c \iint P n_i \, dS = \rho_c (v_{0,t} - 2\epsilon_{jkl}\omega_k v_{0l}) \iint (x_i - x_{di}) n_i \, dS - \rho_c \iint P' n_i \, dS$$

or

$$-\rho_c \iint P n_i \, dS = \tfrac{4}{3}\pi R^3 \rho \,(v_{0i,t} - 2\epsilon_{ijk}\omega_j v_{0k}) - \rho_c \iint P' n_i \, dS, \qquad [30]$$

where we have used the following geometric relationships (Voinov 1973):

$$\iint (x_j - x_{dj})(x_k - x_{dk}) n_i \, dS = 0 \qquad [31]$$

and

$$\iint (x_j - x_{dj}) n_i \, dS = \tfrac{4}{3}\pi R^3 \delta_{ij}. \qquad [32]$$

The second integral on the r.h.s. of [29] can be evaluated by noting the identity, $x_i = x_{di} + (x_i - x_{di})$, and using [31] and [32]. The result is the force due to centripetal acceleration,

$$F_{ci} = -\tfrac{4}{3}\pi R^3 \rho_c \epsilon_{ijk} \omega_j \epsilon_{klm} \omega_l x_{dm}. \qquad [33]$$

The net force on the sphere then becomes

$$-\iint p n_i \, dS = -\rho_c \iint P' n_i \, dS + F_{ci} + \tfrac{4}{3}\pi R^3 \rho_c (v_{0i,t} - 2\epsilon_{ijk}\omega_j v_{0k}) + \tfrac{1}{2}\rho_c \iint v_j v_j n_i \, dS. \qquad [34]$$

In order to evaluate the P' integral, we introduce the vector-valued function Ψ_i, defined by

$$\Psi_{i,jj} = 0, \qquad [35]$$

$$\lim_{(x_i - x_{di}) \to \infty} \Psi_i = 0, \qquad [36]$$

in the region outside the sphere, and satisfying

$$n_j \Psi_{i,j} = n_i \qquad [37]$$

on the surface of the sphere. Using [35], [34] and Green's identity, we have

$$\iint P' n_i \, dS = \iint P' n_j \Psi_{i,j} \, dS = \iint n_j P'_{,j} \Psi_i \, dS. \qquad [38]$$

Then [29] gives

$$\iint P' n_i \, dS = -\iint n_j (\phi_{,jt} - v_{0j,t}) \Psi_i \, dS - \iint 2 n_j \epsilon_{jkl} \omega_k (\phi_{,l} - v_{cl}) \Psi_i \, dS. \qquad [39]$$

These integrals can be evaluated using well-known solutions (Lamb 1932) of [17]–[19] and [35]–[37]:

$$\phi = -v_{di}(x_i - x_{di}) \frac{1}{2} \left\{ \frac{R^3}{[(x_j - x_{dj})(x_j - x_{dj})]^{\frac{3}{2}}} \right\} + (v_{0i} + x_{dj} e_{ij})(x_i - x_{di}) \left\{ 1 + \frac{1}{2} \frac{R^3}{[(x_j - x_{dj})(x_j - x_{dj})]^{\frac{3}{2}}} \right\}$$
$$+ \tfrac{1}{2}(x_i - x_{di}) E_{ij} (x_j - x_{dj}) \left\{ 1 + \frac{2}{3} \frac{R^5}{[(x_k - x_{dk})(x_k - x_{dk})]^{\frac{5}{2}}} \right\} \qquad [40]$$

and

$$\Psi_i = -(x_i - x_{di})\frac{1}{2}\frac{R^3}{[(x_j - x_{dj})(x_j - x_{dj})]^{\frac{3}{2}}}. \quad [41]$$

Using $v_i = \phi_{,i}$, and recognizing that $\phi_{,t} = \phi_{,t}|_{x_{di}} - v_{dj}\phi_{,j}$, we have

$$\iint n_j(\phi_{,jt} - v_{0j,t})\Psi_i \,\mathrm{d}S = \tfrac{2}{3}\pi R^3(v_{0i,t} - v_{di,t}) + \tfrac{2}{3}\pi R^3 v_{dj} e_{ij} \quad [42]$$

and

$$2\epsilon_{jkl}\omega_k \iint n_j(\phi_{,l} - v_{0l})\Psi_i \,\mathrm{d}S = -\tfrac{2}{3}\pi R^3 \epsilon_{jkl}\omega_k(v_{0l} - v_{dl}) - 2\epsilon_{ikl}\omega_k x_{ds} e_{ls}. \quad [43]$$

Similarly, the other integral in [29] becomes

$$\iint v_j v_j n_i \,\mathrm{d}S = \tfrac{4}{3}\pi R^3[3(v_{0l} + x_{dk}e_{lk}) - v_{dl}]e_{il}. \quad [44]$$

Thus, the net force on the sphere is

$$-\iint n_i p \,\mathrm{d}S = \tfrac{2}{3}\pi R^3 \rho_c[v_{0i,t} + (v_{0j} + x_{dk}e_{jk})e_{ji} - v_{di,t}] - \tfrac{2}{3}\pi R^3 \rho_c \epsilon_{ikl}\omega_k(v_{0l} + x_{dj}e_{lj} - v_{dl})$$
$$+ \tfrac{4}{3}\pi R^3 \rho_c[v_{0i,t} + (v_{0j} + x_{dk}e_{jk})e_{ij} - 2\epsilon_{ijk}\omega_j(v_{0k} + x_{ds}e_{ks})] + F_{ci}. \quad [45]$$

It is convenient to note that

$$\tfrac{4}{3}\pi R^3 \rho_c[v_{0i,t} + (v_{0j} + x_{dk}e_{jk})e_{ij} - 2\epsilon_{ijk}\omega_j(v_{0k} + x_{ds}e_{ks})] + F_{ci} = -\tfrac{4}{3}\pi R^3 \mathrm{p}_{0,i}, \quad [46]$$

where p_0 is the pressure in the undisturbed fluid. The names of the other terms in [45] have suffered some confusion. Drew & Lahey (1979) have partitioned these two terms differently and defined a virtual mass force and lift force which were individually objective. In this study we shall call the first term on the l.h.s. of [45] the virtual mass force, and the second term the lift force. If we take the derivative of [9] following the fluid, we have

$$\frac{\mathrm{D_c}v_{ci}}{\mathrm{D}t} = \frac{\partial v_{ci}}{\partial t} + v_{cj}v_{ci,j} = Q_{ij}\frac{\mathrm{D_c}v^*_{cj}}{\mathrm{D}t} + \dot{Q}_{ij}v^*_{cj}$$

or using [6],

$$\frac{\mathrm{D_c}v_{ci}}{\mathrm{D}t} = Q_{ij}\frac{\mathrm{D_c}v^*_{cj}}{\mathrm{D}t} - Q_{ij}\epsilon_{jml}\omega^*_m v^*_{cl}. \quad [47]$$

Thus,

$$\frac{\mathrm{D_c}v_{ci}}{\mathrm{D}t} - \epsilon_{ijk}\omega_j v_{ck} = Q_{ij}\left(\frac{\mathrm{D}_t v^*_{cj}}{\mathrm{D}t} - 2\epsilon_{jkl}\omega^*_k v^*_{cl}\right). \quad [48]$$

For v_{di}, we have from [11],

$$\frac{\mathrm{D_d}v_{di}}{\mathrm{D}t} = Q_{ij}\frac{\mathrm{D_d}v^*_{dj}}{\mathrm{D}t} + 2\dot{Q}_{ij}v^*_{dj} + Q_{ij}x^*_j, \quad [49]$$

so that

$$\frac{\mathrm{D_d}v_{di}}{\mathrm{D}t} - \dot{Q}_{ij}v^*_{dj} - \ddot{Q}_{ij}x^*_j = Q_{ij}\frac{\mathrm{D_d}v^*_{dj}}{\mathrm{D}t} + \dot{Q}_{ij}v^*_{dj}. \quad [50]$$

Differentiating [6] yields

$$\ddot{Q}_{ij} = -\dot{Q}_{ik}\epsilon_{kmj}\omega^*_m, \quad [51]$$

so that

$$-\dot{Q}_{ij}v^*_{dj} - \ddot{Q}_{ij}x^*_j = -\dot{Q}_{ij}(v^*_{dj} + \epsilon_{jkl}\omega^*_j x^*_l) = -\dot{Q}_{ij}Q_{kj}v_{dk}. \quad [52]$$

If we write [11] as

$$v_{d_i} = Q_{ij} v^*_{dj} + \dot{Q}_{ij} x^*_j = Q_{ij} v^*_{dj} + \dot{Q}_{ij} Q_{lj} x_l, \tag{53}$$

we see that

$$v_{di,j} = \dot{Q}_{il} Q_{lj}, \tag{54}$$

so that

$$\frac{D_d v_{di}}{Dt} - \dot{Q}_{ij} v^*_{dj} - Q_{ij} x^*_j = \frac{\partial v_{di}}{\partial t} = Q_{ij}\left(\frac{D_d v^*_{dj}}{Dt} - \epsilon_{jkl}\omega^*_k v^*_{dj}\right). \tag{55}$$

Therefore, in the original starred coordinate system, the combined virtual mass and lift forces become

$$\tfrac{4}{3}\pi R^3 \tfrac{1}{2}\rho \left[\frac{D_c v^*_{cj}}{Dt} - \frac{D_d v^*_{dj}}{Dt} - (v^*_{cj,l} - v^*_{cl,j})(v^*_{cl} - v^*_{dl})\right]. \tag{56}$$

3. INVARIANCE OF THE NONDRAG FORCES

Under any reasonable motion (e.g. nonrelativistic motion) the material flowing is invariant. Cauchy (Truesdell & Toupin 1963) quantified this concept in the Principle of Material Frame Indifference. This principle states that a function which expresses the interaction of a material with itself (e.g. stress or heat flux) should not depend on quantities particular to any one observer. In other words, a constituted variable should depend only on objects. In the same manner that stress and heat flux express the interaction of a material with itself, interfacial heat and momentum transfers express the interaction of one material with another in multiphase flows. Interfacial transfer laws must also be constituted; that is, their dependence on the state variables must be determined. In addition, the dependence of these transfer laws on the state variables must be independent of the frame of reference of the observer.

A fairly general constitutive equation for the nonbuoyant part of the interfacial force on a string has been given previously by Drew & Lahey (1979) as

$$\begin{aligned}\mathbf{M}_d = {} & \alpha B_M(\mathbf{v}_c - \mathbf{v}_d) + \alpha C_{vm}\rho_c\left[\left(\frac{\partial \mathbf{v}_c}{\partial t} + \mathbf{v}_d\cdot\nabla\mathbf{v}_c\right) - \left(\frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_c\cdot\nabla\mathbf{v}_d\right) + (1-\lambda)(\mathbf{v}_d - \mathbf{v}_c)\cdot\nabla(\mathbf{v}_d - \mathbf{v}_c)\right] \\ & + \alpha L_c(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}[\nabla\mathbf{v}_c + (\nabla\mathbf{v}_c)^T] + \alpha L_d(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}[\nabla\mathbf{v}_d + (\nabla\mathbf{v}_d)^T] \\ & + \alpha L_{cd}(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}[\nabla\mathbf{v}_c + (\nabla\mathbf{v}_d)^T] + \alpha L_{dc}(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}[\nabla\mathbf{v}_d + (\nabla\mathbf{v}_c)^T],\end{aligned} \tag{57}$$

where the first term is the interactive drag, and the remainder represent nondrag forces due to temporal and convective accelerations. Equation [57] can be written as

$$\begin{aligned}\mathbf{M}_d = {} & \alpha B_M(\mathbf{v}_c - \mathbf{v}_d) + \alpha C_{vm}\rho_c\left[\left(\frac{\partial \mathbf{v}_c}{\partial t} + \mathbf{v}_c\cdot\nabla\mathbf{v}_c\right) - \left(\frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_d\cdot\nabla\mathbf{v}_d\right)\right] \\ & - \alpha C_{vm}\rho_c(\mathbf{v}_c - \mathbf{v}_d)\cdot\nabla\mathbf{v}_c - \alpha C_{vm}\rho_c(\mathbf{v}_c - \mathbf{v}_d)\cdot\nabla\mathbf{v}_d \\ & - \alpha C_{vm}\rho_c(1-\lambda)(\mathbf{v}_c - \mathbf{v}_d)\cdot\nabla\mathbf{v}_d + \alpha C_{vm}\rho_c(1-\lambda)(\mathbf{v}_c - \mathbf{v}_d)\cdot\nabla\mathbf{v}_c \\ & + \alpha L_c(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}\nabla\mathbf{v}_c + \alpha L_c(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}(\nabla\mathbf{v}_c)^T + \alpha L_d(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}\nabla\mathbf{v}_d + \alpha L_d(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}(\nabla\mathbf{v}_d)^T \\ & + \alpha L_{cd}(\mathbf{v}_c - \mathbf{v}_c)\cdot\tfrac{1}{2}\nabla\mathbf{v}_c + \alpha L_{cd}(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}(\nabla\mathbf{v}_d)^T + \alpha L_{dc}(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}\nabla\mathbf{v}_d + \alpha L_{dc}(\mathbf{v}_d - \mathbf{v}_c)\cdot\tfrac{1}{2}(\nabla\mathbf{v}_c)^T\end{aligned}$$

or

$$\begin{aligned}\mathbf{M}_d = {} & \alpha B_M(\mathbf{v}_c - \mathbf{v}_d) + \alpha C_{vm}\rho_c\left[\left(\frac{\partial \mathbf{v}_c}{\partial t} + \mathbf{v}_c\cdot\nabla\mathbf{v}_c\right) - \left(\frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_d\cdot\nabla\mathbf{v}_d\right)\right] \\ & + (-\alpha C_{vm}\rho_c + \alpha C_{vm}\rho_c(1-\lambda) - \tfrac{1}{2}\alpha L_c - \tfrac{1}{2}\alpha L_{cd})(\mathbf{v}_c - \mathbf{v}_d)\cdot\nabla\mathbf{v}_c \\ & + (-\tfrac{1}{2}\alpha L_c - \tfrac{1}{2}\alpha L_{dc})(\mathbf{v}_c - \mathbf{v}_d)\cdot(\nabla\mathbf{v}_c)^T \\ & + (-\alpha C_{vm}\rho_c - \alpha C_{vm}\rho_c(1-\lambda) - \tfrac{1}{2}\alpha L_d - \tfrac{1}{2}\alpha L_{dc})(\mathbf{v}_c - \mathbf{v}_d)\cdot\nabla\mathbf{v}_d \\ & + (-\tfrac{1}{2}\alpha L_d - \tfrac{1}{2}\alpha L_{cd})(\mathbf{v}_c - \mathbf{v}_d)\cdot(\nabla\mathbf{v}_d)^T.\end{aligned} \tag{58}$$

This expression will have the form of [56], provided that

$$-\alpha C_{vm}\rho_c + \alpha C_{vm}\rho_c(1-\lambda) - \tfrac{1}{2}\alpha L_c - \tfrac{1}{2}\alpha L_{cd} = \alpha L, \quad [59a]$$

$$-\tfrac{1}{2}\alpha L_c - \tfrac{1}{2}\alpha L_{dc} = \alpha L, \quad [59b]$$

$$-\alpha C_{vm}\rho_c - \alpha C_{vm}\rho_c(1-\lambda) - \tfrac{1}{2}\alpha L_d - \tfrac{1}{2}\alpha L_{dc} = 0 \quad [59c]$$

and

$$-\tfrac{1}{2}\alpha L_d - \tfrac{1}{2}\alpha L_{cd} = 0. \quad [59d]$$

Eliminating L_d and L_{dc} from these equations gives

$$-\lambda\alpha C_{vm}\rho_c - \tfrac{1}{2}\alpha L_c - \tfrac{1}{2}\alpha L_{cd} = \alpha L \quad [60a]$$

and

$$-\alpha C_{vm}\rho_c(2-\lambda) + \tfrac{1}{2}\alpha L_{cd} - \alpha L + \tfrac{1}{2}\alpha L_c = 0. \quad [60b]$$

Adding [60a] and [60b] yields,

$$L = -\rho_c C_{vm}. \quad [61]$$

This implies

$$(1-\lambda)\alpha C_{vm}\rho_c = \tfrac{1}{2}\alpha L_c + \tfrac{1}{2}\alpha L_{cd}. \quad [62]$$

There are two degrees of freedom in the problem. If we take $L_{cd} = L_{dc} = 0$, then [59d] implies $L_d = 0$. Thus, [59b] gives

$$L_c = 2L = -2\rho_c C_{vm}. \quad [63]$$

Finally, [60b], [61] and [63] show that

$$(1-\lambda)\alpha C_{vm}\rho_c = -\alpha\rho_c C_{vm}, \quad [64]$$

hence, $\lambda = 2$, which is the correct limit for a single sphere (Drew *et al.* 1979).

The objective form of the interfacial force is then

$$\mathbf{M}_d = \alpha B_M(\mathbf{v}_c - \mathbf{v}_d) + \alpha C_{vm}\rho_c\left[\left(\frac{\partial \mathbf{v}_c}{\partial t} + \mathbf{v}_c\cdot\nabla\mathbf{v}_c\right) - \left(\frac{\partial \mathbf{v}_d}{\partial t} + \mathbf{v}_d\cdot\nabla\mathbf{v}_d\right)\right] - \alpha\rho_c C_{vm}(\mathbf{v}_c - \mathbf{v}_d)\cdot[\nabla\mathbf{v}_c - (\nabla\mathbf{v}_c)^T]. \quad [65]$$

This agrees with [19] if $C_{vm} = 1/2$. Moreover, we note that the virtual mass force can be written as

$$\mathbf{M}_d^{(vm)} = \alpha C_{vm}\rho_c\left(\frac{D_c\mathbf{v}_c}{Dt} - \frac{D_d\mathbf{v}_d}{Dt}\right) \quad [66]$$

and the lift force can be written as

$$\mathbf{M}_d^{(L)} = -\alpha C_{vm}\rho_c(\mathbf{v}_c - \mathbf{v}_d)\cdot[\nabla\mathbf{v}_c - (\nabla\mathbf{v}_c)^T], \quad [67a]$$

or equivalently as

$$\mathbf{M}_d^{(L)} = \alpha C_{vm}\rho_c(\mathbf{v}_c - \mathbf{v}_d)\cdot(\nabla\times\mathbf{v}_c). \quad [67b]$$

It should be stressed that while neither [66] or [67a, b] are objective, their combination in [65] is. For the same case of interest (i.e. the low concentration limit in which $\lambda = 2$), Drew *et al.* (1979) have deduced an objective form of the virtual mass force,

$$\mathbf{M}_d^{(vm)} = -\alpha C_{vm}\rho_c\left[\left(\frac{D_c\mathbf{v}_d}{Dt} - \frac{D_d\mathbf{v}_c}{Dt}\right) + (\mathbf{v}_c - \mathbf{v}_d)\cdot(\nabla\mathbf{v}_d - \nabla\mathbf{v}_c)\right], \quad [68]$$

and an objective "lift force",

$$\mathbf{M}_d^{(L)} = L(\mathbf{v}_d - \mathbf{v}_c)\cdot[\nabla\mathbf{v}_c + (\nabla\mathbf{v}_c)^T]. \quad [69]$$

It is easy to see that the sum of [68] and [69], with $L = \alpha C_{vm}\rho_c$, gives the same result as in [65]. Thus, the previous results yield exactly the same objective interfacial force.

4. CONCLUSION

The force on a single sphere accelerating in an incompressible inviscid fluid, which is accelerating and undergoing constant rotation and strain far from the sphere, has been calculated. The force consists of a virtual mass force, which is one-half the mass of the fluid displaced times the difference in the Lagrangian phasic accelerations, plus a lift force, which is one-half of the mass of the fluid displaced times the fluid rotation tensor dotted with the relative velocity of the sphere with respect to the fluid. The total force is shown to be equal to an objective combination of accelerations, rate of deformation tensors and relative velocities. Thus, the net interfacial drag and nondrag forces are objective.

The solution presented does not show the "classical" Taylor column (Greenspan 1968). The reason for this is that the viscous boundary layer has been implicitly assumed to remain attached to the sphere. If this boundary layer detached, it would lead to shear layers in the flow, resulting in regions where the inviscid equations could have discontinuities. Boundary-layer detachment is expected to occur for sufficiently large rates of rotation. Conjectures have been made that the virtual mass coefficient must depend on rotation rate because the extent of a Taylor column depends on the rotation rate, and the Taylor column also acts as an added mass. We note that the concept of a Taylor column is a quasi-steady one, and its effect on particle accelerations is unproven.

Several researchers, starting with Lord Kelvin, have calculated the force on a sphere moving in an inviscid incompressible fluid. This force is not objective. When the force is calculated using a velocity potential (e.g. Voinov 1973), the fluid far from the sphere is necessarily irrotational. As a consequence, the force calculated in this manner cannot be used to conclude anything about the objectivity (or lack of it) of the interfacial forces. The so-called lateral lift force, which occurs because of the relative motion of a sphere through a rotating fluid (Proudman 1916), is proportional to the rotation tensor of the fluid, and therefore, is not objective either. However, the combination of the virtual mass force and lift force is objective. This is not surprising, since the force on a sphere, $-\iint p\mathbf{n}\,dS$, and the normal, $\mathbf{n}$, are objective. Moreover, static pressure is a physical quantity, and does not depend on the coordinate system used to express it.

In closing, we note that the principle of objectivity, which states that constitutive equations cannot depend in a substantial way on the coordinate systems used to express them, has been fully supported by an exact calculation of the force on a single sphere in an inviscid incompressible fluid. Although the derivation presented herein was done for a single sphere, there is no reason to doubt previous postulates (Drew & Lahey 1979) that for a multiphase mixture the phasic interaction laws must be objective.

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